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936. Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania and Tudorel Lupu (student), Decebal High School, Constanta, Romania.

Let a, b, c be positive real numbers. Prove that

$$a^3 + b^3 + c^3 + 2abc + \frac{9a^2b^2c^2}{(a+b+c)(a^2+b^2+c^2)} \ge ab(a+b) + bc(b+c) + ca(c+a).$$

Solution by Arkady Alt , San Jose ,California, USA.

Assume, due homogeneity, that $a+b+c=1$ and denote $p := ab+bc+ca, q := abc$. Then $a^2+b^2+c^2 = 1-2p, a^3+b^3+c^3 = 1+3q-3p,\ ab(a+b)+bc(b+c)+ca(c+a) = p-3q$, and original inequality becomes

$$1+3q-3p+2q+\frac{9q^2}{1-2p} \ge p-3q \iff 1+8q-4p+\frac{9q^2}{1-2p} \ge 0 \iff$$

(**1**) $9q^2 + 8q(1-2p) - (4p-1)(1-2p) \ge 0.$

We have $p \le \frac{1}{3}$ since $ab+bc+ca \le \frac{(a+b+c)^2}{3}$ and $q \ge \frac{4p-1}{9}$ (Shure's Inequality $\sum_{cyc} a(a-b)(a-c) \ge 0$ in p,q notation and normalized by $a+b+c=1$)

But lower bound $\frac{4p-1}{9}$ for q isn't good enough to prove (**1**).

Therefore, we will try to find a better lower bound for q.

Let $L := \sum_{cyc} a^2b, R := \sum_{cyc} ab^2$ then $L+R = \sum_{cyc} ab(a+b) = p-3q$,

$L \bullet R = \sum_{cyc} a^2b \bullet \sum_{cyc} ab^2 = \sum_{cyc} a^3b^3 + 3a^2b^2c^2 + abc\sum_{cyc} a^3.$

Since $\sum_{cyc} a^3b^3 = p^3+3q^2-3pq$ then $L+R = p^3+3q^2-3pq+$

$3q^2 + q(1+3q-3p) = p^3+9q^2-6pq+q$ and, therefore,

$0 \le (L-R)^2 = (p-3q)^2 - 4(p^3+9q^2-6pq+q) = p^2-6pq+$

$9q^2 - 4p^3 - 36q^2 + 24pq - 4q = p^2 - 4p^3 - 27q^2 - 4q + 18pq \iff$

$\left(q - \frac{9p-2}{27}\right)^2 - \frac{(1-3p)^3}{27} \le 0 \Rightarrow q_* \le q$, where $q_* := \frac{9p-2-2(1-3p)\sqrt{1-3p}}{27}$.

Let $t := \sqrt{1-3p}$ then $p = \frac{1-t^2}{3},\ t \in [0,1] \iff p \in \left[0, \frac{1}{3}\right]$ and $q_* = \frac{(1+t)^2(1-2t)}{27}$.

Since $q \ge 0$, then for $p \in \left[0, \frac{1}{4}\right]$ inequality (1) obviously holds.

If $p \in \left[\frac{1}{4}, \frac{1}{3}\right] \iff t \in \left[0, \frac{1}{2}\right]$ then $1-2p = \frac{1+2t^2}{3}, 4p-1 = \frac{1-4t^2}{3}$ and

$9q^2 + 8q(1-2p) - (4p-1)(1-2p) \ge 9q_*^2 + 8q_*(1-2p) - (4p-1)(1-2p) =$

$\frac{(1+t)^4(1-2t)^2}{81} + \frac{8(1+t)^2(1-2t)(1+2t^2)}{81} - \frac{(1-4t^2)(1+2t^2)}{9} =$

$\frac{t^2(2t-1)^2(t-2)^2}{81} \ge 0.$

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